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ON ONE METHOD FOR OBTAINING UNIQUE SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR AN INTEGRO-DIFFERENTIAL EQUATION

Abstract. The modified method of parametrization is used to study a linear Fredholm integro-differential equation with a degenerate kernel. Using the fundamental matrix, the conditions are established for the existence of a solution to the special Cauchy problem for the Fredholm integro-differential equation with a degenerate kernel. A system of linear algebraic equations is constructed with respect to the introduced additional parameters. Conditions for the unique solvability of a linear boundary value problem for the Fredholm integro-differential equation with a degenerate kernel are obtained.

Key words: boundary value problem, integro-differential equation, unique solvability, fundamental matrix, special Cauchy problem, parametrization method.

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ОБ ОДНОМ СПОСОБЕ ПОЛУЧЕНИЯ ОДНОЗНАЧНОЙ РАЗРЕШИМОСТИ КРАЕВОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

Аннотация. Модифицированный метод параметризации используется для исследования линейного интегро-дифференциального уравнения Фредгольма с вырожденным ядром. С помощью фундаментальной матрицы устанавливаются условия существования решения специальной задачи Коши для интегро-дифференциального уравнения Фредгольма с вырожденным ядром. По введенным дополнительным параметрам построена система линейных алгебраических уравнений. Получены условия однозначной разрешимости линейной краевой задачи для интегро-дифференциального уравнения Фредгольма с вырожденным ядром.

Ключевые слова: краевая задача, интегро-дифференциальное уравнение, однозначная разрешимость, фундаментальная матрица, специальная задача Коши, метод параметризации.

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ИНТЕГРО-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ШЕТТІК ЕСЕПТІҢ БІРМӘНДІ ШЕШІМДІЛІГІН АЛУДЫҢ БІР ӘДІСІ ТУРАЛЫ

Аңдатпа. Модификацияланған параметрлеу әдісі Фредгольмның сызықты интегро-дифференциалдық теңдеуін бұзылған ядромен зерттеу үшін қолданылады. Фундаменталды матрицаның көмегімен бұзылған ядросы бар фредгольмның интегро-дифференциалдық теңдеуі үшін арнайы Коши мәселесін шешудің шарттары белгіленеді. Енгізілген қосымша параметрлер бойынша сызықтық алгебралық теңдеулер жүйесі құрылды. Фредгольмның бұзылған ядросы бар интегро-дифференциалдық теңдеуі үшін сызықтық шекаралық есептің біржақты шешілу шарттары алынды.

Кілт сөздер: шекаралық есеп, интегро-дифференциалдық теңдеу, бір мәнді шешімділік, іргелі матрица, арнайы Коши есебі, параметрлеу әдісі.

Introduction.

We consider the Fredholm integro-differential equation with a degenerate kernel

$$\frac{dx}{dt} = A(t)x + \varphi(t)\int_0^T \psi(\tau)x(\tau)d\tau + f(t), \qquad t \in [0,T], \quad x \in R^n \tag{1}$$

with boundary condition

$$Bx(0) + Cx(T) = d, \quad d \in R^n, \tag{2}$$

where the matrices $A(\mathrm{t}), \varphi(\mathrm{t}), \psi(\mathrm{t})$ and the vector function $f(\mathrm{t})$ are continuous on $[0,T]$.

The study of various physical phenomena leads to the need to study integro-differential equations. Many processes such as bends of beams on elastic foundations [1], vibrations of shafts [2], and bridges [3], the passage of electric currents in inductively coupled circuits [4], plane unsteady motion of a viscous fluid [5], and other processes in which aftereffects are taken into account lead to ordinary integro-differential equations.

Boundary value problems for integro-differential equations have been studied by many authors [6-10]. In the works [11-13] of D.S. Dzhumabaev, the necessary and sufficient conditions for the unique solvability of boundary value problems for integro-differential equations are obtained using the basic parameterization method, where the solution values at the initial points of the partition intervals of the segment $[0,T]$ are used as additional parameters. In this case, for the unknown functions of the auxiliary problem, the initial data appear and, for fixed values of the parameters, they are determined as solutions to the Cauchy problems on their intervals.

Using the boundary conditions and the conditions for gluing the solution at the points of the segment partition, a system of equations for the unknown parameters is compiled. If solutions to the Cauchy problems exist for a wide class of right-hand sides of systems of differential equations, then the solvability of the systems of equations with respect to the introduced parameters requires the fulfillment of conditions for the existence of theorems. The applicability of the known theorems depends on the type and properties of the constructed systems of equations. Since the form of this system depends not only on the right-hand side of the differential equation and the boundary condition, but also on the method of introducing additional parameters, the choice of additional parameters is often crucial for the successful study and solution of the considered boundary value problem.

In this article, a two-point boundary value problem for systems of Fredholm integro-differential equations with a degenerate kernel is studied by a modified parameterization method, where additional parameters are introduced as solution values in the midpoints of the partition intervals of the segment $[0,T]$. Using this method, we can use the initial data to establish new conditions for the existence of a solution, expand the classes of solvable two-point boundary value problems, and propose constructive algorithms for finding their solutions.

By Δ_1 we denote the case where there is no partition of the interval $[0,T]$. Let $C([0,T],\Delta_1,R^n)$ be the space of continuous functions $x:[0,T]\to R^n$ on $[0,T]$ with norm $\|x\|_1=\max_{t\in[0,T]}\|x(t)\|$. We introduce the notation for the value of the function $x(t)$ at the point $t=\frac{T}{2}$ by λ, that is, $\lambda \hat{=} x\left(\frac{T}{2}\right)$ and change $u=x(t)-\lambda$ on the interval $[0,T]$. Then problem (1) - (2) is reduced to an equivalent boundary value problem with the parameter:

$$\frac{du}{dt}=A(t)(u+\lambda)+\varphi(t)\int_0^T\psi(\tau)(u(\tau)+\lambda)d\tau+f(t),\qquad t\in[0,T],\quad u\left(\frac{T}{2}\right)=0 \tag{3}$$

$$Bu(0)+B\lambda+C\lim_{t\to T-0}u(t)+C\lambda=d,\quad d\in R^n \tag{4}$$

Using the fundamental matrix $X(t)$ of the ordinary differential equation $\frac{dx}{dt}=A(t)x,\quad t\in[0,T]$, we can obtain a solution to the special Cauchy problem (3) for fixed values of the parameter λ:

$$u(t,\lambda)=X(t)\int_{\frac{T}{2}}^{t}X^{-1}(\tau)\left\{A(\tau)\lambda+\varphi(\tau)\int_0^T\psi(\tau_1)(u(\tau_1)+\lambda)d\tau_1+f(\tau)\right\}d\tau,\quad t\in[0,T].$$

(5)

Consider the Cauchy problems on the subintervals

$$\frac{dz}{dt} = A(t)z + P(t), \qquad z\left(\frac{T}{2}\right) = 0, \quad t \in \left[0, \frac{T}{2}\right] \tag{6}$$

$$\frac{dz}{dt} = A(t)z + P(t), \qquad z\left(\frac{T}{2}\right) = 0, \quad t \in \left[\frac{T}{2}, T\right] \tag{7}$$

where $P(t)-$ is a continuous on $[0,T]$ square function or vector of dimension n. By $a(p,t,\xi_1)$, $a(p,t,\xi_2)$ we denote the unique solution to the Cauchy problem (3) on each interval. The uniqueness of the solution to the Cauchy problem for linear ordinary differential equations implies that

$$a(p,t,\xi_1) = X(t)\int_{T/2}^{t} X^{-1}(\tau)P(\tau)d\tau, \quad t \in \left[0, \frac{T}{2}\right] \tag{8}$$

$$a(p,t,\xi_2) = X(t)\int_{T/2}^{t} X^{-1}(\tau)P(\tau)d\tau, \quad t \in \left[\frac{T}{2}, T\right]. \tag{9}$$

Therefore, we can represent Δ_1 the general solution of equation (1) in the form:

$$x(\Delta_1, t, \lambda) = \lambda + a(A,t,\xi_1)\lambda + a(f,t,\xi_1), \qquad t \in \left[0, \frac{T}{2}\right] \tag{10}$$

$$x(\Delta_1, t, \lambda) = \lambda + a(A,t,\xi_2)\lambda + a(f,t,\xi_2), \qquad t \in \left[\frac{T}{2}, T\right] \tag{11}$$

Let's introduce the notation:

$$\mu = \int_0^T \psi(t)u(t)dt = \int_0^{T/2} \psi(t)u(t)dt + \int_{T/2}^{T} \psi(t)u(t)dt. \tag{12}$$

In equality (12), instead of $u(t)$, substituting the right-hand side of (5), we obtain

$$\mu = \int_0^{T/2} \psi(t)X(t)\int_{\frac{T}{2}}^{t} X^{-1}(\tau)\left\{A(\tau)\lambda + \varphi(\tau)\int_0^T \psi(\tau_1)(u(\tau_1) + \lambda)d\tau_1 + f(\tau)\right\}d\tau dt +$$

$$+ \int_{T/2}^{T} \psi(t)X(t)\int_{\frac{T}{2}}^{t} X^{-1}(\tau)\left\{A(\tau)\lambda + \varphi(\tau)\int_0^T \psi(\tau_1)(u(\tau_1) + \lambda)d\tau_1 + f(\tau)\right\}d\tau dt. \tag{13}$$

Taking into account the solution of the Cauchy problem (8), (9), from equality (13) we obtain:

$$\mu[I - G(\Delta_1)] = V(\Delta_1)\lambda + g(\Delta_1), \tag{14}$$

where $G(\Delta_1) = \int_0^{T/2} \psi(t)a(\varphi,t,\xi_1)dt + \int_{T/2}^{T} \psi(t)a(\varphi,t,\xi_2)dt,$

$$V(\Delta_1) = \int_0^{T/2} \psi(t)\left[a(A,t,\xi_1) + a(\varphi,t,\xi_1)\int_0^T \psi(\tau_1)d\tau_1\right]d\tau +$$

$$+ \int_{T/2}^{T} \psi(t)\left[a(A,t,\xi_2) + a(\varphi,t,\xi_2)\int_0^T \psi(\tau_1)d\tau_1\right]dt,$$

$$g(\Delta_1)=\int_0^{T/2}\psi(t)a(f,t,\xi_1)dt+\int_{T/2}^{T}\psi(t)a(f,t,\xi_2)dt,$$

where $I-$ is the identity matrix.

The following theorem is true.

Theorem 1. *The special Cauchy problem (3), (4) for the Fredholm integro-differential equation is solvable if and only if the matrix* $I-G(\Delta_1)$ *is invertible.*

Solvability of the boundary value problem. Let's introduce the notation: $M(\Delta_1)=[I-G(\Delta_1)]^{-1}$. Then it follows from (14) that

$$\mu=M(\Delta_1)V(\Delta_1)\lambda+M(\Delta_1)g(\Delta_1). \qquad (15)$$

Taking into account (8), (9) and (12), the integral equation (5) will be rewritten as:

$$u(t)=a(A,t,\xi)\lambda+a(\varphi,t,\xi)\left[\mu+\int_0^T\psi(\tau)d\tau\lambda\right]+a(f,t,\xi) \qquad (16)$$

Substituting the value μ, i.e. the right-hand side of (15) into equality (16) we obtain

$$u(t)=\alpha(t)\lambda+\beta(t), \qquad (17)$$

where $$\alpha(t)=a(\varphi,t,\xi)\left[M(\Delta_1)V(\Delta_1)+\int_0^T\psi(\tau)d\tau\right]+a(A,t,\xi),$$

$$\beta(t)=a(\varphi,t,\xi)M(\Delta_1)g(\Delta_1)+a(f,t,\xi).$$

From (17), determining $\lim\limits_{t\to T-0}u(t)$ and substituting it into the boundary conditions (4), we obtain a linear system of equations with respect to the introduced parameter

$$Q_*(\Delta_1)\cdot\lambda=-F_*(\Delta_1) \qquad (18)$$

where $Q_*(\Delta_1)-$ is a matrix with elements: $Q_*(\Delta_1)=B(\alpha(0)+I)+C(\alpha(T)+I)$ and $F_*(\Delta_1)-$ vector function $F_*(\Delta_1)=-d+B\beta(0)+C\beta(T)$.

Definition. *Problem (1), (2) is called uniquely solvable if for any pair* $(f(t),d),\quad f(t)\in C([0,T],\Delta_1,R^n),\ d\in R^n$, *it has a unique solution.*

Theorem 2. If the matrix $I-G(\Delta_1)$ is invertible, the boundary value problem (1), (2) is uniquely solvable if and only if the matrix $Q_*(\Delta_1)$ is invertible.

Example. We consider the Fredholm integro-differential equation with a degenerate kernel

$$\frac{dx}{dt}=A(t)x+\varphi(t)\int_0^1\psi(\tau)x(\tau)d\tau+f(t),\qquad t\in[0,1],\quad x\in R^2 \qquad (19)$$

with boundary condition

$$Bx(0)+Cx(1)=d,\quad d\in R^2, \qquad (20)$$

where the matrices $A(\mathrm{t}),\varphi(\mathrm{t}),\psi(\mathrm{t})$ and the vector function $f(t)$ are continuous on $[0,1]$.

$$T=1,\quad A=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad \varphi=\begin{pmatrix}2&0\\0&2\end{pmatrix},\quad \psi(s)=\begin{pmatrix}s&0\\1&1\end{pmatrix},\quad f(t)=\begin{pmatrix}-2/3\\2t-11/3\end{pmatrix},$$

$$B=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad C=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad d=\begin{pmatrix}3\\1\end{pmatrix}$$

In this problem, the identity matrix $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ - is the fundamental matrix of the differential part.

By Δ_1 we denote the case when there is no partition of the interval $[0,1]$.

We introduce the notation for the value of the function $x(t)$ at the point $t=\frac{1}{2}$ by λ, i.e. $\lambda \hat{=} x\left(\frac{1}{2}\right)$ and make the change $u(t)=x(t)-\lambda$ on the interval [0,1].

Then problem (1) - (2) is reduced to an equivalent boundary value problem with a parameter:

$$\frac{du}{dt}=A(t)(u+\lambda)+\varphi(t)\int_0^1\psi(\tau)(u(\tau)+\lambda)d\tau+f(t), \qquad t\in[0,1], \quad u\left(\frac{1}{2}\right)=0$$

(21)

$$Bu(0)+B\lambda+C\lim_{t\to 1-0}u(t)+C\lambda=d, \quad d\in R^2$$

(22)

The use of the fundamental matrix $X(t)$ of the ordinary differential equation $\frac{dx}{dt}=A(t)x, \quad t\in[0,1]$ makes it possible to obtain a solution to the special Cauchy problem for fixed values of the parameter λ:

$$u(\Delta_1,\xi,t)=X(t)\int_{\frac{1}{2}}^{t}X^{-1}(\tau)\left\{A(\tau)\lambda+\varphi(\tau)\int_0^1\psi(\tau_1)(u(\tau_1)+\lambda)d\tau_1+f(\tau)\right\}d\tau, \quad t\in[0,1].$$

(23)

Consider the Cauchy problems on the subintervals

$$\frac{dz}{dt}=A(t)z+P(t), \qquad z\left(\frac{1}{2}\right)=0, \quad t\in\left[0,\frac{1}{2}\right]$$

(24)

$$\frac{dz}{dt}=A(t)z+P(t), \qquad z\left(\frac{1}{2}\right)=0, \quad t\in\left[\frac{1}{2},1\right]$$

(25)

where $P(t)$ is a square function or a vector of dimension n continuous on $[0,1]$. By $a(p,\xi,t)$, we denote the unique solution of the Cauchy problem (3) on each interval. The uniqueness of the solution to the Cauchy problem for linear ordinary differential equations implies that

$$a(p,\xi,t)=X(t)\int_{1/2}^{t}X^{-1}(\tau)P(\tau)d\tau, \quad t\in\left[0,\frac{1}{2}\right]$$

(26)

$$a(p,\xi,t)=X(t)\int_{1/2}^{t}X^{-1}(\tau)P(\tau)d\tau, \quad t\in\left[\frac{1}{2},1\right].$$

(27)

Therefore, we can represent Δ_1 the general solution of equation (1) in the form:

$$x(\Delta_1, t, \lambda) = \lambda + a(A, t, \xi)\lambda + a(f, t, \xi), \qquad t \in \left[0, \frac{1}{2}\right]$$

(28)

$$x(\Delta_1, t, \lambda) = \lambda + a(A, t, \xi)\lambda + a(f, t, \xi), \qquad t \in \left[\frac{1}{2}, 1\right]$$

(29)

Let's introduce the notation:

$$\mu = \int_0^1 \psi(t) u(t)\, dt = \int_0^{1/2} \psi(t) u(t)\, dt + \int_{1/2}^1 \psi(t) u(t)\, dt. \tag{30}$$

In equality (12), instead of $u(t)$, substituting the right-hand side of (5), we obtain

$$\mu = \int_0^{1/2} \psi(t) X(t) \int_{\frac{1}{2}}^{t} X^{-1}(\tau) \left\{ A(\tau)\lambda + \varphi(\tau) \int_0^1 \psi(\tau_1)\big(u(\tau_1) + \lambda\big) d\tau_1 + f(\tau) \right\} d\tau dt +$$

$$+ \int_{1/2}^{1} \psi(t) X(t) \int_{\frac{1}{2}}^{t} X^{-1}(\tau) \left\{ A(\tau)\lambda + \varphi(\tau) \int_0^1 \psi(\tau_1)\big(u(\tau_1) + \lambda\big) d\tau_1 + f(\tau) \right\} d\tau dt$$

(31)

Taking into account the solution of the Cauchy problem (8), (9), from equality (13) we obtain:

$$\mu\left[I - G(\Delta_1, \xi)\right] = V(\Delta_1, \xi)\lambda + g(\Delta_1, \xi), \tag{32}$$

$$G(\Delta_1, \xi) = \int_0^{1/2} \psi(t) a(\varphi, t, \xi)\, dt + \int_{1/2}^1 \psi(t) a(\varphi, t, \xi)\, dt,$$

where

$$V(\Delta_1, \xi) = \int_0^{1/2} \psi(t) \left[a(A, t, \xi_1) + a(\varphi, t, \xi_1) \int_0^1 \psi(\tau_1) d\tau_1 \right] d\tau +$$

$$+ \int_{1/2}^1 \psi(t) \left[a(A, t, \xi) + a(\varphi, t, \xi) \int_0^1 \psi(\tau_1) d\tau_1 \right] dt,$$

$$g(\Delta_1, \xi) = \int_0^{1/2} \psi(t) a(f, t, \xi)\, dt + \int_{1/2}^1 \psi(t) a(f, t, \xi)\, dt,$$

where $I-$ is the identity matrix. Calculate the elements of the matrix $\left[I - G(\Delta_1, \xi)\right]$:

$$I - G(\Delta_1, \xi) = I - \int_0^{0.5} \psi(t) a(\varphi, \xi, t) dt - \int_{0.5}^1 \psi(t) a(\varphi, \xi, t) dt =$$

$$= I - \int_0^{0.5} \psi(t) X(t) \int_{0.5}^{t} X^{-1}(\tau) \varphi(\tau) d\tau dt - \int_{0.5}^{1} \psi(t) X(t) \int_{0.5}^{t} X^{-1}(\tau) \varphi(\tau) d\tau dt =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \int_0^{1/2} \begin{pmatrix} t & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \int_{1/2}^{t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} d\tau dt - \int_{1/2}^{1} \begin{pmatrix} t & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \int_{1/2}^{t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} d\tau dt =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \int_0^{1/2} \begin{pmatrix} t & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2t-1 & 0 \\ 0 & 2t-1 \end{pmatrix} dt - \int_{1/2}^{1} \begin{pmatrix} t & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2t-1 & 0 \\ 0 & 2t-1 \end{pmatrix} dt =$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\int_0^{1/2}\begin{pmatrix}t & 0\\1 & 1\end{pmatrix}\begin{pmatrix}2t-1 & 0\\0 & 2t-1\end{pmatrix}dt-\int_{1/2}^{1}\begin{pmatrix}t & 0\\1 & 1\end{pmatrix}\begin{pmatrix}2t-1 & 0\\0 & 2t-1\end{pmatrix}dt=$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\int_0^{1/2}\begin{pmatrix}2t^2-t & 0\\2t-1 & 2t-1\end{pmatrix}dt-\int_{1/2}^{1}\begin{pmatrix}2t^2-t & 0\\2t-1 & 2t-1\end{pmatrix}dt=$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}\frac{2}{3}t^3-\frac{1}{2}t^2 & 0\\t^2-t & t^2-t\end{pmatrix}\Bigg|_0^{\frac{1}{2}}-\begin{pmatrix}\frac{2}{3}t^3-\frac{1}{2}t^2 & 0\\t^2-t & t^2-t\end{pmatrix}\Bigg|_{\frac{1}{2}}^{1}=$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}-\frac{1}{24} & 0\\-\frac{1}{4} & -\frac{1}{4}\end{pmatrix}-\begin{pmatrix}\frac{5}{24} & 0\\\frac{1}{4} & \frac{1}{4}\end{pmatrix}=\begin{pmatrix}5/6 & 0\\0 & 1\end{pmatrix}$$

$$\mu=M(\Delta_1)V(\Delta_1,\xi)\lambda+M(\Delta_1)g(\Delta_1,\xi). \tag{33}$$

Then $M(\Delta_1)=\left[I-G(\Delta_1,\xi)\right]^{-1}=\frac{6}{5}\begin{pmatrix}1 & 0\\0 & 5/6\end{pmatrix}=\begin{pmatrix}6/5 & 0\\0 & 1\end{pmatrix}$. Therefore the matrix $M(\Delta_1)$ is invertible. We calculate the matrix $V(\Delta_1,\xi)$ and the vector function $g(\Delta_1,\xi)$:

$$V(\Delta_1,\xi)=\int_0^{1/2}\psi(t)a(\varphi,\xi,t)\int_0^1\psi(\tau)d\tau dt+\int_{1/2}^{1}\psi(t)a(\varphi,\xi,t)\int_0^1\psi(\tau)d\tau dt+$$

$$+\int_0^{1/2}\psi(t)a(A,\xi,t)dt+\int_{1/2}^{1}\psi(t)a(A,\xi,t)dt=\int_0^{1/2}\psi(t)X(t)\int_{1/2}^{t}X^{-1}(\tau)\varphi(\tau)\int_0^1\psi(\tau_1)d\tau_1 d\tau dt+$$

$$+\int_{1/2}^{1}\psi(t)X(t)\int_{1/2}^{t}X^{-1}(\tau)\varphi(\tau)\int_0^1\psi(\tau_1)d\tau_1 d\tau dt+$$

$$+\int_0^{1/2}\psi(t)a(A,\xi,t)dt+\int_{1/2}^{1}\psi(t)a(A,\xi,t)dt=\int_0^{1/2}\begin{pmatrix}t & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\int_0^1\begin{pmatrix}\tau_1 & 0\\1 & 1\end{pmatrix}d\tau_1 d\tau dt+$$

$$+\int_{1/2}^{1}\begin{pmatrix}t & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\int_0^1\begin{pmatrix}\tau_1 & 0\\1 & 1\end{pmatrix}d\tau_1 d\tau dt=$$

$$\int_0^{1/2}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 1 & 1\end{pmatrix}d\tau dt +$$

$$+\int_{1/2}^{1}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 1 & 1\end{pmatrix}d\tau dt =$$

$$=\int_0^{1/2}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix} 1 & 0 \\ 2 & 2\end{pmatrix}d\tau dt + \int_{1/2}^{1}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix} 1 & 0 \\ 2 & 2\end{pmatrix}d\tau dt =$$

$$=\int_0^{1/2}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} t-1/2 & 0 \\ 2t-1 & 2t-1\end{pmatrix}dt + \int_{1/2}^{1}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} t-1/2 & 0 \\ 2t-1 & 2t-1\end{pmatrix}dt =$$

$$=\int_0^{1/2}\begin{pmatrix} t^2-\frac{1}{2}t & 0 \\ 3t-\frac{3}{2} & 2t-1\end{pmatrix}dt + \int_{1/2}^{1}\begin{pmatrix} t^2-\frac{1}{2}t & 0 \\ 3t-\frac{3}{2} & 2t-1\end{pmatrix}dt =$$

$$=\left.\begin{pmatrix} \frac{1}{3}t^3-\frac{t^2}{4} & 0 \\ \frac{3}{2}t^2-\frac{3}{2}t & t^2-t\end{pmatrix}\right|_0^{1/2} + \left.\begin{pmatrix} \frac{1}{3}t^3-\frac{t^2}{4} & 0 \\ \frac{3}{2}t^2-\frac{3}{2}t & t^2-t\end{pmatrix}\right|_{1/2}^{1} = \begin{pmatrix} -\frac{1}{48} & 0 \\ -\frac{3}{8} & -\frac{1}{4}\end{pmatrix} + \begin{pmatrix} \frac{5}{48} & 0 \\ \frac{3}{8} & \frac{1}{4}\end{pmatrix} = \begin{pmatrix} 1/12 & 0 \\ 0 & 0\end{pmatrix},$$

$$g(\Delta_1,\xi) = \int_0^{1/2}\psi(t)a(f,\xi,t)dt + \int_{1/2}^{1}\psi(t)a(f,\xi,t) =$$

$$= \int_0^{1/2}\psi(t)X(t)\int_{1/2}^{t}X^{-1}(\tau)f(\tau)d\tau dt + \int_{1/2}^{1}\psi(t)X(t)\int_{1/2}^{t}X^{-1}(\tau)f(\tau)d\tau dt =$$

$$=\int_0^{1/2}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} -\frac{2}{3} \\ 2\tau-\frac{11}{3}\end{pmatrix}d\tau dt + \int_{1/2}^{1}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\int_{1/2}^{t}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} -\frac{2}{3} \\ 2\tau-\frac{11}{3}\end{pmatrix}d\tau dt =$$

$$=\int_0^{1/2}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\left.\begin{pmatrix} -\frac{2}{3}\tau \\ \tau^2-\frac{11}{3}\tau\end{pmatrix}\right|_{\frac{1}{2}}^{t} dt + \int_{1/2}^{1}\begin{pmatrix} t & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\left.\begin{pmatrix} -\frac{2}{3}\tau \\ \tau^2-\frac{11}{3}\tau\end{pmatrix}\right|_{\frac{1}{2}}^{t} dt =$$

$$=\begin{pmatrix} 1/72 \\ 11/24\end{pmatrix} + \begin{pmatrix} -5/72 \\ 2/24\end{pmatrix} = \begin{pmatrix} -1/18 \\ 1/12\end{pmatrix}$$

Taking into account (8), (9) and (12), the integral equation (5) will be rewritten as:

$$u(\Delta_1,\xi,t) = a(A,\xi,t)\lambda + a(\varphi,\xi,t)\left[\mu + \int_0^1\psi(\tau)d\tau\lambda\right] + a(f,\xi,t) \quad (34)$$

Substituting the μ value, i.e. the right-hand side of (15) into equality (16) we obtain

$$u\left(\Delta_1,\xi,t\right)=\alpha\left(\Delta_1,\xi,t\right)\lambda+\beta\left(\Delta_1,\xi,t\right), \tag{35}$$

where $\alpha\left(\Delta_1,\xi,t\right)=a\left(\varphi,\xi,t\right)\left[M\left(\Delta_1\right)V\left(\Delta_1\right)+\int_0^1\psi(\tau)d\tau\right]+a\left(A,\xi,t\right)$,

$$\beta\left(\Delta_1,\xi,t\right)=a\left(\varphi,\xi,t\right)M\left(\Delta_1\right)g\left(\Delta_1,\xi\right)+a\left(f,\xi,t\right).$$

From (17), determining $\lim\limits_{t\to 1-0}u\left(\Delta_1,\xi,t\right)$ and substituting it into the boundary conditions (4), we obtain a linear system of equations with respect to the introduced parameter

$$Q_*\left(\Delta_1,\xi\right)\cdot\lambda=-F_*\left(\Delta_1,\xi\right) \tag{36}$$

where $Q_*\left(\Delta_1,\xi\right)-$ is a matrix with elements:

$Q_*\left(\Delta_1,\xi\right)=B\left(\alpha\left(\Delta_1,\xi,0\right)+I\right)+C\left(\alpha\left(\Delta_1,\xi,\mathrm{T}\right)+I\right)$ and $F_*\left(\Delta_1,\xi\right)-$ vector function

$F_*\left(\Delta_1,\xi\right)=-d+B\beta\left(\Delta_1,\xi,0\right)+C\beta\left(\Delta_1,\xi,\mathrm{T}\right)$.

Let's carry out preliminary calculations

$$a(\varphi,\xi,0)=X(0)\int_{1/2}^{0}X^{-1}(\tau)\varphi(\tau)d\tau=\begin{pmatrix}1&0\\0&1\end{pmatrix}\int_{1/2}^{0}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}d\tau=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

$$a(\varphi,\xi,1)=X(1)\int_{1/2}^{1}X^{-1}(\tau)\varphi(\tau)d\tau=\begin{pmatrix}1&0\\0&1\end{pmatrix}\int_{1/2}^{1}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}d\tau=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$a(A,\xi,0)=0,\quad a(A,\xi,1)=0$$

$$a\left(f,\xi,0\right)=X\left(0\right)\int_{1/2}^{0}X^{-1}\left(\tau\right)f\left(\tau\right)d\tau=\begin{pmatrix}1&0\\0&1\end{pmatrix}\int_{1/2}^{0}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}-2/3\\2t-11/3\end{pmatrix}dt=\begin{pmatrix}1/3\\19/12\end{pmatrix}$$

$$a\left(f,\xi,1\right)=X\left(1\right)\int_{1/2}^{1}X^{-1}\left(\tau\right)f\left(\tau\right)d\tau=\begin{pmatrix}1&0\\0&1\end{pmatrix}\int_{1/2}^{1}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}-2/3\\2t-11/3\end{pmatrix}dt=\begin{pmatrix}-1/3\\-13/12\end{pmatrix}$$

$$a(\varphi,\xi,0)\int_0^1\psi(t)dt=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\int_0^1\begin{pmatrix}t&0\\1&1\end{pmatrix}dt=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}\frac{t^2}{2}&0\\t&t\end{pmatrix}\Bigg|_0^1=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}\frac{1}{2}&0\\1&1\end{pmatrix}=\begin{pmatrix}-1/2&0\\-1&-1\end{pmatrix}$$

$$a(\varphi,\xi,1)\int_0^1\psi(t)dt=\begin{pmatrix}1&0\\0&1\end{pmatrix}\int_0^1\begin{pmatrix}t&0\\1&1\end{pmatrix}dt=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{t^2}{2}&0\\t&t\end{pmatrix}\Bigg|_0^1=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{1}{2}&0\\1&1\end{pmatrix}=\begin{pmatrix}1/2&0\\1&1\end{pmatrix}$$

$$\alpha\left(\Delta_1,\xi,0\right)=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\left[\begin{pmatrix}6/5&0\\0&1\end{pmatrix}\begin{pmatrix}1/12&0\\0&0\end{pmatrix}+\begin{pmatrix}1/2&0\\1&1\end{pmatrix}\right]=\begin{pmatrix}-3/5&0\\-1&-1\end{pmatrix}$$

$$\alpha\left(\Delta_1,\xi,1\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\left[\begin{pmatrix}6/5&0\\0&1\end{pmatrix}\begin{pmatrix}1/12&0\\0&0\end{pmatrix}+\begin{pmatrix}1/2&0\\1&1\end{pmatrix}\right]=\begin{pmatrix}3/5&0\\1&1\end{pmatrix}$$

$$\beta\left(\Delta_1,\xi,0\right)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}6/5 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1/18\\ 1/12\end{pmatrix}+\begin{pmatrix}1/3\\ 19/12\end{pmatrix}=\begin{pmatrix}1/15\\ -1/12\end{pmatrix}+\begin{pmatrix}5/15\\ 19/12\end{pmatrix}=\begin{pmatrix}2/5\\ 3/2\end{pmatrix}$$

$$\beta\left(\Delta_1,\xi,1\right)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}6/5 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1/18\\ 1/12\end{pmatrix}+\begin{pmatrix}-1/3\\ -13/12\end{pmatrix}=\begin{pmatrix}-1/15\\ 1/12\end{pmatrix}+\begin{pmatrix}-5/15\\ -13/12\end{pmatrix}=\begin{pmatrix}-2/5\\ -1\end{pmatrix}$$

Let's calculate the elements of the matrix $Q_*\left(\Delta_1,\xi\right)$:

$$B\Big(\alpha\left(\Delta_1,\xi,0\right)+I\Big)+C\Big(\alpha\left(\Delta_1,\xi,1\right)+I\Big)=$$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}-3/5 & 0\\ -1 & -1\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}3/5 & 0\\ 1 & 1\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$$

Let us calculate the element of the $F_*\left(\Delta_1,\xi\right)$ vector function

$$-d+B\beta\left(\Delta_1,\xi,0\right)+C\beta\left(\Delta_1,\xi,1\right)=-\begin{pmatrix}3\\ 1\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2/5\\ 3/2\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-2/5\\ -1\end{pmatrix}$$

Let's calculate the value of λ:

$$\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\cdot\lambda=\begin{pmatrix}3\\ 1/2\end{pmatrix}\qquad \lambda=\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\cdot\begin{pmatrix}3\\ 1/2\end{pmatrix}=\begin{pmatrix}3/2\\ 1/4\end{pmatrix}$$

Substituting the found value λ in (15), we calculate μ:

$$\mu=\begin{pmatrix}6/5 & 0\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}1/12 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}3/2\\ 1/4\end{pmatrix}+\begin{pmatrix}-1/18\\ 1/12\end{pmatrix}\right]=$$

$$=\begin{pmatrix}6/5 & 0\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}1/8\\ 0\end{pmatrix}+\begin{pmatrix}-1/18\\ 1/12\end{pmatrix}\right]=\begin{pmatrix}6/5 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}5/72\\ 1/12\end{pmatrix}=\begin{pmatrix}1/12\\ 1/12\end{pmatrix}$$

$$u(\Delta_1,\xi,t)=X(t)\int_{\frac{1}{2}}^{t}X^{-1}(\tau)\left\{A(\tau)\lambda+\varphi(\tau)\int_0^1\psi(\tau_1)\big(u(\tau_1)+\lambda\big)d\tau_1+f(\tau)\right\}d\tau,\quad t\in[0,1]$$

$$Y^*(t)=A(\tau)\lambda+\varphi(\tau)\int_0^1\psi(\tau_1)\big(u(\tau_1)+\lambda\big)d\tau_1+f(\tau)$$

$$Y^*(t)=\varphi(\tau)\left[\mu+\int_0^1\psi(s)ds\lambda\right]+f(t)$$

$$Y^*(t)=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\left[\begin{pmatrix}1/12\\ 1/12\end{pmatrix}+\int_0^1\begin{pmatrix}s & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}3/2\\ 1/4\end{pmatrix}ds\right]+\begin{pmatrix}-2/3\\ 2t-11/3\end{pmatrix}=$$

$$=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\left[\begin{pmatrix}1/12\\ 1/12\end{pmatrix}+\begin{pmatrix}3/4\\ 7/4\end{pmatrix}\right]+\begin{pmatrix}-2/3\\ 2t-11/3\end{pmatrix}=$$

$$=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}5/6\\ 11/6\end{pmatrix}+\begin{pmatrix}-2/3\\ 2t-11/3\end{pmatrix}=\begin{pmatrix}5/3\\ 11/3\end{pmatrix}+\begin{pmatrix}-2/3\\ 2t-11/3\end{pmatrix}=\begin{pmatrix}1\\ 2t\end{pmatrix}$$

$$u^*(\Delta_1,\xi,t)=X(t)\int_{\frac{1}{2}}^{t}X^{-1}(\tau)Y^*(\tau)\,d\tau,$$

$$x^*(t)=\lambda+u^*(\Delta_1,\xi,t)$$

$$x^*(t)=\begin{pmatrix}3/2\\ 1/4\end{pmatrix}+\int_{1/2}^{t}\begin{pmatrix}1\\ 2t\end{pmatrix}d\tau=\begin{pmatrix}3/2\\ 1/4\end{pmatrix}+\begin{pmatrix}t\\ t^2\end{pmatrix}\Bigg|_{1/2}^{t}=\begin{pmatrix}3/2\\ 1/4\end{pmatrix}+\begin{pmatrix}t-1/2\\ t^2-1/4\end{pmatrix}=\begin{pmatrix}t+1\\ t^2\end{pmatrix}$$

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